COL7160 : Quantum Computing

Lecture 2: Qubits

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1 Mathematical Notation

We represent vectors over $\mathbb{C}^d$ using the *braket*[1] or the *Dirac* notation. A *"ket"* $|\cdot\rangle$ represents a d-dimensional column vector in the vector space over complex number $\mathbb{C}^d$:

$$|v\rangle = \begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_{d-1} \end{pmatrix}$$

$$\langle v| = \left(v_0^*, v_1^*, \dots, v_{d-1}^*\right)$$

Similarly, a *"bra"* represents a d-dimensional row vector equal to the complex conjugate of the corresponding ket. We can keep this in mind by writing $\langle\cdot| = (|\cdot\rangle^*)^\top$, or equvalently by writing $\langle\cdot| = |\cdot\rangle^\dagger$. Here, $\dagger$ is called the *Hermitian dagger* and it represents the conjugate transpose. For any vector $|u\rangle \in \mathbb{C}^d$ we have $\langle u| = (|u\rangle^*)^\top = |u\rangle^\dagger$. For matrices we write $A^\dagger = (A^*)^\top$.

An *inner product* on the complex vector space $\mathbb{C}^d$ is a map $\langle\cdot|\cdot\rangle : \mathbb{C}^d \times \mathbb{C}^d \to \mathbb{C}$ satisfying the following properties for all $|u\rangle, |v\rangle, |w\rangle \in \mathbb{C}^d$ and all $\alpha \in \mathbb{C}$:

1. $\langle u|v\rangle = \langle v|u\rangle^*$
2. $\langle u|\alpha v + w\rangle = \alpha \langle u|v\rangle + \langle u|w\rangle$.
3. $\langle v|v\rangle \geq 0,\quad$ with equality if and only if $|v\rangle = 0$.

Let $|v_1\rangle = (a_1, \dots, a_d)^\top$ and $|v_2\rangle = (b_1, \dots, b_d)^\top$. Then the *standard inner product* on $\mathbb{C}^d$ is defined by

$$\langle v_1|v_2\rangle = \sum_{i=1}^{d} a_i^* b_i.$$

This can be equivalently written as $\langle v_1|v_2\rangle = |v_1\rangle^\dagger |v_2\rangle$. Using the inner product, we can define the length or *norm* of a ket $|v\rangle = (v_0, \dots, v_d)^\top$ as $\| \, |v\rangle \, \|_2 := \sqrt{\langle v|v\rangle} = \sqrt{\sum_{i=1}^d v_i^* v_i} = \sqrt{\sum_{i=1}^d |v_i|^2}$. if $\| \, |v\rangle \, \| = 1$, we say that $|v\rangle$ is normalized.

Example 1. If $|v\rangle = (1-i, 0)^\top$ and $|w\rangle = (2i, 3)^\top$ then $\langle v|w\rangle = -2 + 2i$.

Example 2. If $|v\rangle = \frac{1}{2}(1+i, 1-i)^\top$ then $\| \, |v\rangle \, \|_2 = 1$.

We will use the notation $|j\rangle \in \mathbb{C}^d$ for $j \in \{0, 1, \dots, d-1\}$ to denote the *computational basis vectors*, defined by $|j\rangle = (0, \dots, 0, 1, 0, \cdots, 0)^\top$ where the entry 1 appears in the j-th position and all other entries are zero. Notice that $\{|0\rangle, |1\rangle, \dots, |d-1\rangle\}$ forms an orthonormal basis of $\mathbb{C}^d$, satisfying $\langle i|j\rangle = \delta_{ij}$. Hence, every vector $|v\rangle \in \mathbb{C}^d$ can be written uniquely as a linear combination of these basis vectors:

$$|v\rangle = \sum_{j=0}^{d-1} v_j \, |j\rangle \, ,$$

where $v_j = \langle j|v\rangle$ denotes the j-th coordinate of $|v\rangle$ in the computational basis.

[1] Recall that in high-school physics we used the $\vec{\cdot}$ to denote vectors such as $\vec{F} = m\vec{a}$ or $\vec{E} = \frac{q}{4\pi\varepsilon_0} \frac{\vec{r}}{\|\vec{r}\|^3}$. The braket notation can be thought of as a fancy way of doing the same thing

2 Quantum Bits

Classically, a bit takes values in $\{0, 1\}$. We represent classical bits using vectors in $\mathbb{C}^2$ by identifying

$$0 \longleftrightarrow |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad 1 \longleftrightarrow |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

A *quantum bit* or *qubit* can be in any state of the form

$$|\psi\rangle = \alpha |0\rangle + \beta |1\rangle ,$$

where $\alpha, \beta \in \mathbb{C}$ satisfy

$$|\alpha|^2 + |\beta|^2 = 1.$$

The complex numbers α and β are called the *amplitudes* of the qubit.

Thus, a qubit is a normalized vector in $\mathbb{C}^2$. We therefore identify the *state space of a qubit* with $\mathbb{C}^2$.

Example 3. The states

$$|+\rangle := \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \qquad |-\rangle := \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

are valid qubit states.

Notice that $|0\rangle$ and $|1\rangle$ are orthonormal and form a basis for $\mathbb{C}^2$. This basis is called the *standard basis* or the *computational basis* of $\mathbb{C}^2$. There are many other orthonormal bases for $\mathbb{C}^2$. One important example is the *Hadamard basis*

$$\mathcal{H} = \{|+\rangle , |-\rangle\}.$$

Example 4. Verify that $|+\rangle$ and $|-\rangle$ form an orthonormal basis of $\mathbb{C}^2$.

Exercise 1. Express $|1\rangle$ in the Hadamard basis, i.e., find α, β such that

$$|1\rangle = \alpha |+\rangle + \beta |-\rangle .$$

3 Multiple Qubits

Classically, the state of two bits is given by a string in $\{00, 01, 10, 11\}$. Quantum mechanically, the state of two qubits lives in the vector space

$$\mathbb{C}^2 \otimes \mathbb{C}^2 \cong \mathbb{C}^4.$$

The computational basis for two qubits is

$$\{|00\rangle , |01\rangle , |10\rangle , |11\rangle\},$$

where

$$|00\rangle == \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad |01\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad |10\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad |11\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

A *pure state* of two qubits is any normalized vector of the form

$$|\psi\rangle = \alpha_{00} |00\rangle + \alpha_{01} |01\rangle + \alpha_{10} |10\rangle + \alpha_{11} |11\rangle ,$$

where $\alpha_{ij} \in \mathbb{C}$ and

$$\sum_{i,j \in \{0,1\}} |\alpha_{ij}|^2 = 1.$$

More generally, a pure state of n qubits is a unit vector in

$$(\mathbb{C}^2)^{\otimes n} \cong \mathbb{C}^{2^n},$$

and can be written as

$$|\psi\rangle = \sum_{x\in\{0,1\}^n} \alpha_x |x\rangle, \qquad \sum_{x\in\{0,1\}^n} |\alpha_x|^2 = 1.$$

The vectors $\{|x\rangle : x \in \{0,1\}^n\}$ form the computational basis for n qubits.

Tensor Product

If

$$|\psi\rangle = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_m \end{pmatrix}, \qquad |\varphi\rangle = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_n \end{pmatrix},$$

then their tensor product is

$$|\psi\rangle \otimes |\varphi\rangle = \begin{pmatrix} \psi_1\varphi_1 \\ \psi_1\varphi_2 \\ \vdots \\ \psi_1\varphi_n \\ \psi_2\varphi_1 \\ \vdots \\ \psi_m\varphi_n \end{pmatrix} \in \mathbb{C}^{mn}.$$

Let

$$|\psi_A\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle, \qquad |\psi_B\rangle = \beta_0 |0\rangle + \beta_1 |1\rangle.$$

The joint state of the two systems is given by the *tensor product*

$$|\psi_A\rangle \otimes |\psi_B\rangle = \alpha_0\beta_0 |00\rangle + \alpha_0\beta_1 |01\rangle + \alpha_1\beta_0 |10\rangle + \alpha_1\beta_1 |11\rangle.$$

If $|u\rangle \in \mathbb{C}^{d_1}$ and $|v\rangle \in \mathbb{C}^{d_2}$, then

$$|u\rangle \otimes |v\rangle \in \mathbb{C}^{d_1 d_2}.$$

The tensor product satisfies the following properties:

1. **Distributivity:**
$$|u\rangle \otimes (|v\rangle + |w\rangle) = |u\rangle \otimes |v\rangle + |u\rangle \otimes |w\rangle.$$

2. **Associativity:**
$$(|u\rangle \otimes |v\rangle) \otimes |w\rangle = |u\rangle \otimes (|v\rangle \otimes |w\rangle).$$

Note that, in general,

$$|u\rangle \otimes |v\rangle \neq |v\rangle \otimes |u\rangle,$$

so the tensor product is *not commutative.*

Entanglement

Example 5 (EPR / Bell state)**.** The state

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

is called an *EPR pair* or a *Bell state*.

Example 6. Let

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|+\rangle\,|+\rangle + |-\rangle\,|-\rangle).$$

Then

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle),$$

which is an EPR pair.

Exercise 2. Show that the EPR state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ cannot be written as a tensor product of two single-qubit states.

References